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VIBRATIONS IN A PARAMETRICALLY EXCITED SYSTEM

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This paper deals with vibrations of parametrically excited non-linear systems with one degree of freedom. The non-linearity is cubic and is of the same order as the linear terms. The parametric vibrations are excited by a periodical force of Jacobi elliptic type. The mathematical model of the system is a special type of non-linear Hill's equation. The analytical approximate solution of the equation is obtained applying the elliptic-Krylov–Bogolubov method (method of variable phase and amplitude) developed for strong non-linear differential equation of Duffing type. It enables the regions of unbounded solution to be defined approximately. The parameters of a dynamic absorber which transforms the motion to regular are calculated in this paper.

1. INTRODUCTION

The problem of parametrically excited systems has been known for a long time. A significant number of papers have been published concerning the problem. All the papers can be divided into two main groups: the first, in which the motion and the boundary between stable and unstable solutions are defined, and the second, in which the methods for transforming motion to regular one are considered. It is impossible to find the closed-form solution of the parametrically excited system described with a linear second order differential equation with periodical time variable coefficients. Many approximate analytical and numerical methods have been developed (see references [1–5]). For all of them it is common that they tend to define the regions of stable and unstable motion caused by parametrical resonance. The transverse motion of a straight beam with a uniform cross-section loaded by an axial time-varying force is described with the second order non-linear differential equation with time variable parameters [1, 2, 6–9]. For the case where the non-linearity is small, perturbation methods such as the method of Bogolubov–Mitropolski [1], the method of multiple scales [2], and the method of normal forms [10] have been developed for solving the problems. In papers [11–13], the parametrically excited pendulum has been considered. This pendulum has been shown to exhibit equilibrium, oscillating, rotating and tumbling motions which can be periodic or chaotic. It is shown that by using small perturbation on the system, the chaotic behavior may be changed into regular motion which can flexibly be chosen from a variety of unstable periodic orbits with different desired responses.

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In this paper, parametrically excited strong non-linear cubic system is considered. The parameter variation is formulated by a Jacobi elliptic function. For solving the equation, the elliptic-Krylov–Bogolubov method is applied. The approximate procedure is suggested for obtaining the regions of bounded and unbounded solutions. In this paper, the method for transforming the unbounded motion to a periodic one by applying a dynamic absorber is shown. The parameters of the linear dynamic absorber are defined.

2. PARAMETRIC RESONANCE—NON-LINEAR CASE

The mathematical model of a strong non-linear parametrically excited system is a non-linear differential equation with periodic coefficient

$$\ddot{y}_1 + [\delta - \varepsilon F(t)]y_1 - c_3^* y_1^3 = 0, \tag{1}$$

where c_3^* is the coefficient of non-linear term, δ and ε are constants and $(\ddot{\ }) = \mathrm{d}^2/\mathrm{d}t^2$. Equation (1) is the non-linear Hill's equation.

For the case where the periodic function is the sinus Jacobi elliptic function and the amplitude of excitation is F_0 it is

$$F(t) = F_0 \,\mathrm{sn}(\Omega t, k^2), \tag{2}$$

where Ω is the frequency of excitation and k is the modulus of the elliptic function. The Hill's equation (1) has the form

$$\ddot{y}_1 + \omega^2[1 - h\,\mathrm{sn}(\Omega t, k^2)]y_1 - c_3^* y_1^3 = 0, \tag{3}$$

where

$$\varepsilon^* = \varepsilon F_0, \quad \omega^2 = \delta, \quad h = \varepsilon^*/\omega^2. \tag{4}$$

Equation (3) describes the parametric-excited vibrations.

Let us assume that the parameter h is small. Then the elliptic-Krylov–Bogolubov procedure developed for strong non-linear differential equations [14–18] can be applied. It represents an analytical approximate procedure based on the perturbation of the solution of the strong non-linear differential equation. For $h = 0$, equation (3) is transformed to the strong non-linear differential equation with cubic non-linearity of Duffing type. The well-known solution is

$$y_1 = A\,\mathrm{sn}(\Omega_1 t + \theta, k_1^2), \tag{5}$$

where

$$\Omega_1^2 = \omega^2 - \frac{c_3^* A^2}{2}, \quad k_1^2 = \frac{c_3^* A^2}{2\Omega_1^2}, \tag{6}$$

and A and θ are constants determined by the initial conditions.

For the resonant case

$$\Omega_1 \approx \frac{\Omega}{2}, \quad k \approx k_1, \tag{7}$$

the trial solution of equation (3) is the same as the form of its generating solution (5). The trial solution is given by equation (3) but with A and θ now time dependent

$$y_1 = A(t)\,\mathrm{sn}(\psi, k^2), \tag{8}$$

where

$$\psi = \frac{\Omega t}{2} + \theta(t). \tag{9}$$

The task of finding the solution of y_1 is transformed into finding two functions $A(t)$ and $\theta(t)$. The assumed solution has to satisfy the constraint that the time derivative of the trial solution must have the same form as the time derivative of the generating solution

$$\dot{y}_1 = A\frac{\Omega}{2}\,\mathrm{cn}(\psi, k^2)\,\mathrm{dn}(\psi, k^2). \tag{10}$$

Differentiating equation (8) with respect to t and using relation (10) one has

$$\dot{A}\,\mathrm{sn}(\psi, k^2) + \dot{\theta}\,\mathrm{cn}(\psi, k^2)\,\mathrm{dn}(\psi, k^2) = 0. \tag{11}$$

Differentiating equation (10), and substituting the results into equation (3) one gets

$$\dot{A}\frac{\Omega}{2}\,\mathrm{cn}(\psi, k^2)\,\mathrm{dn}(\psi, k^2) - A\frac{\Omega}{2}\,\dot{\theta}\,\mathrm{sn}(\psi, k^2)[1 - 2k^2\,\mathrm{sn}^2(\psi, k^2) + k^2]$$

$$= h\omega^2 A\,\mathrm{sn}(\psi, k^2)\,\mathrm{sn}[2(\psi - \theta), k^2]. \tag{12}$$

The sn function with double argument is (see reference [19])

$$\mathrm{sn}[2(\psi - \theta), k^2] = 2\frac{\mathrm{sn}[(\psi - \theta), k^2]\,\mathrm{cn}[(\psi - \theta), k^2]\,\mathrm{dn}[(\psi - \theta), k^2]}{1 - k^2\,\mathrm{sn}^4[(\psi - \theta), k^2]}$$

$$= 2\frac{1 - k^2\,\mathrm{sn}^2\psi\,\mathrm{sn}^2\theta}{(1 - k^2\,\mathrm{sn}^2\psi\,\mathrm{sn}^2\theta)^4 - k^2\,\mathrm{sn}\,\psi\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta - \mathrm{cn}\,\psi\,\mathrm{dn}\,\psi\,\mathrm{sn}\,\theta)^4}$$

$$\times(\mathrm{sn}\,\psi\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta - \mathrm{cn}\,\psi\,\mathrm{dn}\,\psi\,\mathrm{sn}\,\theta)(\mathrm{cn}\,\psi\,\mathrm{cn}\,\theta + \mathrm{sn}\,\psi\,\mathrm{dn}\,\psi\,\mathrm{sn}\,\theta\,\mathrm{dn}\,\theta)$$

$$\times(\mathrm{dn}\,\psi\,\mathrm{dn}\,\theta + k^2\,\mathrm{sn}\,\psi\,\mathrm{cn}\,\psi\,\mathrm{sn}\,\theta\,\mathrm{cn}\,\theta). \tag{13}$$

Substituting equations (11) and (13) into equation (12), two first order differential equations are obtained:

$$A\dot{\theta}(1 - k^2 \operatorname{sn}^4 \psi) i = -\frac{2}{\Omega} bh\omega^2 A \operatorname{sn} \psi^2, \tag{14}$$

$$\dot{A}(1 - k^2 \operatorname{sn}^4 \psi) i = \frac{2}{\Omega} bh\omega^2 A \operatorname{sn} \psi \operatorname{cn} \psi \operatorname{dn} \psi, \tag{15}$$

i.e.,

$$A\dot{\theta} = -\frac{2}{\Omega}\frac{b}{i}\frac{\operatorname{sn}^2 \psi}{(1 - k^2 \operatorname{sn}^4 \psi)} h\omega^2 A, \tag{16}$$

$$\dot{A} = \frac{2}{\Omega}\frac{\operatorname{sn} \psi \operatorname{cn} \psi \operatorname{dn} \psi}{(1 - k^2 \operatorname{sn}^4 \psi)}\frac{b}{i} h\omega^2 A, \tag{17}$$

where

$$\operatorname{sn} \psi \equiv \operatorname{sn}(\psi, k^2), \quad \operatorname{sn} \theta \equiv \operatorname{sn}(\theta, k^2),$$

$$\operatorname{cn} \psi \equiv \operatorname{cn}(\psi, k^2), \quad \operatorname{cn} \theta \equiv \operatorname{cn}(\theta, k^2),$$

$$\operatorname{dn} \psi \equiv \operatorname{dn}(\psi, k^2), \quad \operatorname{dn} \theta \equiv \operatorname{dn}(\theta, k^2),$$

$$\begin{aligned} b = {} & \operatorname{sn} \psi \operatorname{cn} \psi \operatorname{dn} \psi \operatorname{cn}^2 \theta \operatorname{dn}^2 \theta + k^2 \operatorname{sn}^3 \psi \operatorname{cn} \psi \operatorname{dn}^3 \psi \operatorname{sn}^4 \theta \operatorname{dn}^2 \theta \\ & - k^4 \operatorname{sn}^5 \psi \operatorname{cn} \psi \operatorname{dn} \psi \operatorname{cn}^2 \theta \operatorname{dn}^2 \theta \operatorname{sn}^4 \theta - k^2 \operatorname{sn} \psi \operatorname{cn}^3 \psi \operatorname{dn} \psi \operatorname{sn}^2 \theta \operatorname{cn}^2 \theta \\ & + k^4 \operatorname{sn}^3 \psi \operatorname{cn}^3 \psi \operatorname{dn} \psi \operatorname{sn}^4 \theta \operatorname{cn}^2 \theta - \operatorname{sn} \psi \operatorname{cn} \psi \operatorname{dn}^3 \psi \operatorname{sn}^2 \theta \operatorname{dn}^2 \theta \\ & + k^2 \operatorname{sn}^2 \psi \operatorname{cn}^2 \psi \operatorname{cn}^3 \theta \operatorname{dn} \theta \operatorname{sn} \theta + \operatorname{sn}^2 \psi \operatorname{dn}^2 \psi \operatorname{cn} \theta \operatorname{dn}^3 \theta \operatorname{sn} \theta \\ & - k^2 \operatorname{sn}^4 \psi \operatorname{dn}^2 \psi \operatorname{cn} \theta \operatorname{dn}^3 \theta \operatorname{sn}^3 \theta - \operatorname{cn}^2 \psi \operatorname{dn}^2 \psi \operatorname{sn} \theta \operatorname{cn} \theta \operatorname{dn} \theta \\ & + k^4 \operatorname{sn}^4 \psi \operatorname{cn}^2 \psi \operatorname{dn}^2 \psi \operatorname{sn}^5 \theta \operatorname{dn} \theta \operatorname{cn} \theta - k^4 \operatorname{sn}^4 \psi \operatorname{cn}^2 \psi \operatorname{cn}^3 \theta \operatorname{dn} \theta \operatorname{sn}^3 \theta, \end{aligned}$$

and

$$\begin{aligned} i = {} & 1 - 4k^2 \operatorname{sn}^2 \psi \operatorname{sn}^2 \theta - 4k^6 \operatorname{sn}^6 \psi \operatorname{sn}^6 \theta + k^8 \operatorname{sn}^8 \psi \operatorname{sn}^8 \theta - k^2 \operatorname{sn}^4 \psi \operatorname{cn}^4 \theta \operatorname{dn}^4 \theta \\ & + 6k^4 \operatorname{sn}^4 \psi \operatorname{sn}^4 \theta - k^2 \operatorname{cn}^4 \psi \operatorname{dn}^4 \psi \operatorname{sn}^4 \theta - 6k^2 \operatorname{sn}^2 \psi \operatorname{cn}^2 \psi \operatorname{dn}^2 \psi \operatorname{sn}^2 \theta \operatorname{cn}^2 \theta \operatorname{dn}^2 \theta \\ & + 4k^2 \operatorname{sn}^3 \psi \operatorname{cn} \psi \operatorname{dn} \psi \operatorname{cn}^8 \theta \operatorname{dn}^3 \theta \operatorname{sn} \theta + 4k^2 \operatorname{sn} \psi \operatorname{cn}^3 \psi \operatorname{dn}^3 \psi \operatorname{cn} \theta \operatorname{dn} \theta \operatorname{sn}^3 \theta. \end{aligned}$$

The task of obtaining solution y_1 of equation (3) is transformed into equivalent one of obtaining two solutions $A(t)$ and $\theta(t)$ of the system of equation (16) and (17). It is not possible to find the solutions of the equations (16) and (17) in the closed form. As the elliptic functions are periodical with period 4K, where K is the total elliptical integral of the first kind, the averaging process suggested by Yuste and Bejarano [18] is applied:

$$\dot{\theta} = \left(\Omega_1 - \frac{\Omega}{2}\right) - \frac{2}{\Omega}\frac{h\omega^2}{4\mathrm{K}}\int_0^{4\mathrm{K}} \frac{b}{i}\frac{\mathrm{sn}^2\,\psi}{(1 - k^2\,\mathrm{sn}^4\,\psi)}\,\mathrm{d}\psi, \tag{18}$$

$$\dot{A} = \frac{2}{\Omega}\frac{h\omega^2 A}{4\mathrm{K}}\int_0^{4\mathrm{K}} \frac{\mathrm{sn}\,\psi\,\mathrm{cn}\,\psi\,\mathrm{dn}\,\psi}{(1 - k^2\,\mathrm{sn}^4\,\psi)}\frac{b}{i}\,\mathrm{d}\psi. \tag{19}$$

The averaging procedure is shown in the Appendix A.

To prove the exactness of the procedure the linear system as a special case will be considered.

2.1. LINEAR CASE

For the linear case

$$k = 0, \quad \mathrm{cn} = \cos, \quad \mathrm{sn} = \sin, \quad \mathrm{dn} = 1, \quad \mathrm{K}(0) = \frac{\pi}{2}.$$

Equations (16) and (17) are transformed to

$$\dot{\theta} = -\frac{2}{\Omega}h\omega^2 \sin 2\theta \frac{1}{2\pi}\int_0^{2\pi} (\sin^4\psi = \sin^2\psi\cos^2\psi)\,\mathrm{d}\psi,$$

$$\dot{A} = \frac{4}{\Omega}h\omega^2 A\cos 2\theta \frac{1}{2\pi}\int_0^{2\pi} \sin^2\psi\cos^2\psi\,\mathrm{d}\psi,$$

i.e.,

$$\dot{\theta} = \left(\omega - \frac{\Omega}{2}\right) - \frac{h\omega^2}{2\Omega}\sin 2\theta, \tag{20}$$

$$\dot{A} = \frac{h\omega^2 A}{2\Omega}\cos 2\theta. \tag{21}$$

These equations are the same as those obtained in reference [1]. It also proves the correctness of the method discussed in this paper as well as its generality.

3. BOUNDED AND UNBOUNDED SOLUTIONS

For parametrically excited vibrations it is evident that small excitation may produce large response. It is of special interest to separate the regions of stable and unstable motion. The aim of this section is to define approximately the first unstable region.

Consider equations (14) and (15). Averaging separately the left- and the right-hand sides of the equations gives

$$\int_0^{4K} i(1-k^2\,\mathrm{sn}^4\,\psi\,\mathrm{d}\psi) = (1-k^2\,\mathrm{sn}^4\,\theta)\int_0^{4K}(1-k^2\,\mathrm{sn}^4\,\psi)^2(1-k^2\,\mathrm{sn}^2\,\theta\,\mathrm{sn}^2\,\psi)^2\,\mathrm{d}\psi, \quad (22)$$

$$\begin{aligned}\int_0^{4K} b\,\mathrm{sn}^2\,\psi\,\mathrm{d}\psi &= -\,\mathrm{sn}\,\theta\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta\Bigg[\int_0^{4K}\mathrm{sn}^2\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi\\ &\quad + k^2\,\mathrm{sn}^2\,\theta\,\mathrm{dn}^2\,\theta\int_0^{4K}\mathrm{sn}^6\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi\\ &\quad + k^4\,\mathrm{sn}^4\,\theta\int_0^{4K}\mathrm{sn}^6\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi\\ &\quad - k^4\,\mathrm{cn}^2\,\theta\,\mathrm{sn}^2\,\theta\int_0^{4K}\mathrm{sn}^6\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{d}\psi\Bigg]\\ &= -\frac{1}{2}\,\mathrm{sn}\,2\theta(1-k^2)(1-k^2\,\mathrm{sn}^4\,\theta)\int_0^{4K}\\ &\quad \times\mathrm{sn}^4\,\psi(1+\mathrm{sn}^2\,\theta k^2\,\mathrm{sn}^2\,\psi)\,\mathrm{d}\psi, \end{aligned} \quad (23)$$

and

$$\begin{aligned}\int_0^{4K} b\,\mathrm{sn}\,\psi\,\mathrm{cn}\,\psi\,\mathrm{dn}\,\psi\,\mathrm{d}\psi &= \mathrm{cn}^2\,\theta\,\mathrm{dn}^2\,\theta\int_0^{4K}\mathrm{sn}^2\,\psi\,\mathrm{cn}^2\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi\\ &\quad - \mathrm{sn}^2\,\theta\,\mathrm{dn}^2\,\theta\int_0^{4K}\mathrm{sn}^2\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{dn}^4\,\psi\,\mathrm{d}\psi\\ &\quad + k^2\,\sin^4\,\theta\,\mathrm{dn}^2\,\theta\int_0^{4K}\mathrm{sn}^4\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{dn}^4\,\psi\,\mathrm{d}\psi\\ &\quad - k^4\,\mathrm{cn}^2\,\theta\,\mathrm{dn}^2\,\theta\,\mathrm{sn}^4\,\theta\int_0^{4K}\mathrm{sn}^6\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi\\ &\quad - k^2\,\mathrm{sn}^2\,\theta\,\mathrm{cn}^2\,\theta\int_0^{4K}\mathrm{sn}^2\,\psi\,\mathrm{cn}^4\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi\\ &\quad + k^4\,\mathrm{sn}^4\,\theta\,\mathrm{cn}^2\,\theta\int_0^{4K}\mathrm{sn}^4\,\psi\,\mathrm{cn}^4\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi\\ &= \mathrm{cn}\,2\theta(1-k^2)(1-k^2\,\mathrm{sn}^4\,\theta)\int_0^{4K}\mathrm{sn}^4\,\psi\,(1-k^2\,\mathrm{sn}^2\,\psi\,\mathrm{sn}^2\,\theta)\end{aligned}$$

$$\times (1 - 2k^2 \operatorname{sn}^2 \theta + k^2 \operatorname{sn}^2 \psi \operatorname{sn}^2 \theta)$$

$$+ (1 - k^2) \int_0^{4K} \operatorname{sn}^4 \psi (1 - k^2 \operatorname{sn}^2 \psi \operatorname{sn}^2 \theta)$$

$$\times \{- 2k^2 \operatorname{sn}^4 \theta \operatorname{dn} 2\theta + k^2 \operatorname{sn}^2 \theta \operatorname{cn} 2\theta)]$$

$$+ k^2 \operatorname{sn}^2 \psi \operatorname{sn}^2 \theta (1 - k^2 \operatorname{sn}^4 \theta) [\operatorname{dn} 2\theta + 1]\} \, d\psi.$$

Then equations (14) and (15) transform to

$$\dot{A} \frac{1}{4K} \int_0^{4K} (1 - k^2 \operatorname{sn}^4 \psi)^2 (1 - k^2 \operatorname{sn}^2 \theta \operatorname{sn}^2 \psi)^2 \, d\psi$$

$$= \frac{2}{\Omega} h\omega^2 A (1 - k^2) \frac{1}{4K} \int_0^{4K} \operatorname{sn}^4 \psi (1 - k^2 \operatorname{sn}^2 \psi \operatorname{sn}^2 \theta)$$

$$\times \left\{ \operatorname{cn} 2\theta - 2k^2 \operatorname{sn}^2 \theta \left[\operatorname{cn} 2\theta + \operatorname{sn}^2 \theta (\operatorname{dn} 2\theta + k^2 \operatorname{sn}^2 \theta \operatorname{cn} 2\theta) \frac{1}{1 - k^2 \operatorname{sn}^4 \theta} \right] \right.$$

$$\left. + k^2 \operatorname{sn}^2 \psi \operatorname{sn}^2 \theta (1 - k^2 \operatorname{sn}^4 \theta) [\operatorname{cn} 2\theta + \operatorname{dn} 2\theta + 1] \right\} d\psi, \tag{24}$$

$$A \dot{\theta} \frac{1}{4K} \int_0^{4K} (1 - k^2 \operatorname{sn}^4 \psi)^2 (1 - k^2 \operatorname{sn}^2 \theta \operatorname{sn}^2 \psi)^2 \, d\psi$$

$$= \frac{1}{\Omega} h\omega^2 A \frac{\operatorname{sn} 2\theta}{4K} (1 - k^2) \int_0^{4K} \operatorname{sn}^4 \psi (1 + \operatorname{sn}^2 \theta k^2 \operatorname{sn}^2 \psi) \, d\psi. \tag{25}$$

For the case where the vibrations are small, only the first terms of the equation are considered. Then, the simplified equations (24) and (25) are

$$\dot{\theta} = \left(\Omega_1 - \frac{\Omega}{2} \right) + \frac{h\omega^2}{\Omega} \frac{\sin 2\theta}{4K} (1 - k^2) \int_0^{4K} \operatorname{sn}^4 \psi \, d\psi, \tag{26}$$

$$\dot{A} = \frac{2h\omega^2 A}{\Omega} \frac{\operatorname{cn} 2\theta}{4K} (1 - k^2) \int_0^{4K} \operatorname{sn}^4 \psi \, d\psi, \tag{27}$$

where

$$\int_0^{4K} \operatorname{sn}^4 \psi \, d\psi \equiv A_4 = \frac{4}{3k^4} [(2 + k^2)\mathrm{K} - 2(1 + k^2)\mathrm{E}], \tag{28}$$

with E being the total elliptic integral of the second kind. Equations (26) and (27) are transformed by introducing two new variables

$$x = A\,\mathrm{sn}\,\theta, \quad y = A\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta. \tag{29}$$

The amplitude A and phase θ are calculated from equation (29):

$$A^2 = \frac{y^2 + (1+k^2)x^2 \pm \sqrt{[y^2 + (1+k^2)x^2]^2 - 4k^2x^4}}{2}, \tag{30}$$

$$\theta = \mathrm{sn}^{-1}\left(\frac{x}{A}, k^2\right). \tag{31}$$

Substituting equations (26) and (27) into equation (29) gives

$$\dot{x} = p[2 - (3+k^2)\,\mathrm{sn}^2\,\theta + 2k^2\,\mathrm{sn}^4\,\theta]\,x + \left(\Omega_1 - \frac{\Omega}{2}\right)y, \tag{32}$$

$$\dot{y} = py[1 - \mathrm{sn}^2(3 + k^2 - 3k^2\,\mathrm{sn}^2\,\theta] - x[1 + k^2 - k^2\,\mathrm{sn}^2\,\theta]\left(\Omega_1 - \frac{\Omega}{2}\right). \tag{33}$$

Linearizing relations (32) and (33) gives

$$\dot{x} = 2px + \left(\Omega_1 - \frac{\Omega}{2}\right)y, \tag{34}$$

$$\dot{y} = py - x(1+k^2)\left(\Omega_1 - \frac{\Omega}{2}\right), \tag{35}$$

where

$$p = \frac{2h\omega^2}{\Omega}\,\frac{1-k^2}{4\mathrm{K}}\,A_4. \tag{36}$$

The properties of the solutions of the system of equations (34) and (35) depend on the solutions of the characteristic equation

$$\begin{vmatrix} \lambda - 2p & -(\Omega_1 - \frac{\Omega}{2}) \\ (1+k^2)(\Omega_1 - \frac{\Omega}{2}) & \lambda - p \end{vmatrix} = 0. \tag{37}$$

The characteristic exponents are

$$\lambda_{1,2} = \frac{3p \pm \sqrt{p^2 - 4(\Omega_1 - \Omega/2)^2(1+k^2)}}{2}. \tag{38}$$

The solution of the equations are

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \tag{39}$$

$$y = C_1 \frac{\lambda_1 - 2p}{\Omega_1 - \Omega/2} e^{\lambda_1 t} + C_2 \frac{\lambda_2 - 2p}{\Omega_1 - \Omega/2} e^{\lambda_2 t}, \tag{40}$$

where C_1 and C_2 are arbitrary constants. The amplitude of vibrations is bounded and stable with time if the real value of λ or real part λ is zero or negative. It is satisfied for

$$p \leqslant 0, \quad p^2 - 4\left(\Omega_1 - \frac{\Omega}{2}\right)^2 (1 + k^2) \leqslant 0. \tag{41}$$

When the real part of λ is positive definite, the amplitude of vibrations A is unbounded and the motion is unstable in time. The conditions of instability are

$$p > 0, \quad p > \left| 2\left(\Omega_1 - \frac{\Omega}{2}\right)\sqrt{1 + k^2} \right|. \tag{42}$$

The first condition is satisfied for the case where the modulus of elliptic function is

$$0 < k^2 < 0{\cdot}63. \tag{43}$$

Using equation (36), relation (42) becomes

$$\frac{h\omega^2}{2\Omega_1} \frac{1 - k^2}{4\mathrm{K}} A_4 > \left| \left(\Omega_1 - \frac{\Omega}{2}\right)\sqrt{1 + k^2} \right|. \tag{44}$$

It means that if the frequency of excitation is in the interval

$$2\Omega_1 - \frac{1}{\sqrt{1 + k^2}} \frac{h\omega^2}{\Omega_1} \frac{1 - k^2}{4\mathrm{K}} A_4 < \Omega < 2\Omega_1 + \frac{1}{\sqrt{1 + k^2}} \frac{h\omega^2}{\Omega_1} \frac{1 - k^2}{4\mathrm{K}} A_4, \tag{45}$$

the amplitude of vibration increases. Relation (45) defines approximately one of the regions of instability. It not only depends on the properties of the system, but also on the initial amplitude.

Introducing the dimensionless parameters

$$\omega^* = \frac{2\omega}{\Omega}, \quad \Omega^* = 1 - \frac{c_3^{*2} A^2}{2\omega^2}, \tag{46}$$

the boundaries of the unstable regions are described as

$$\omega^* \sqrt{\Omega^*} = 1 \pm \frac{h}{\sqrt{1 + k^2}} \frac{1 - k^2}{8K\Omega^*} A_4. \tag{47}$$

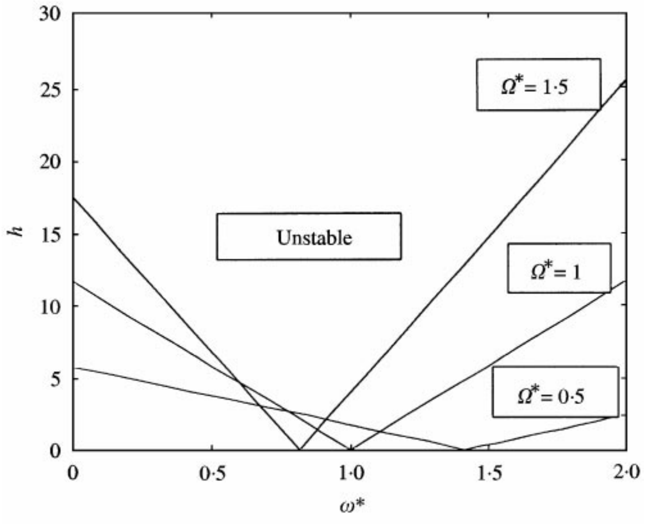


Figure 1. h–ω^* diagrams for $k^2 = 0{\cdot}5$ and various values of Ω^*.

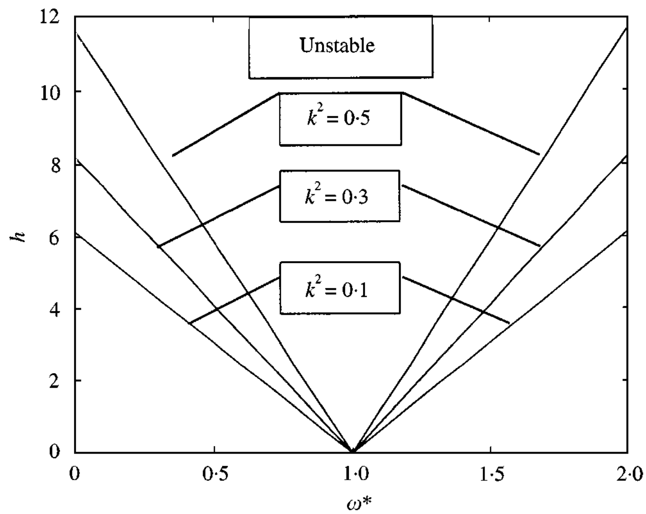
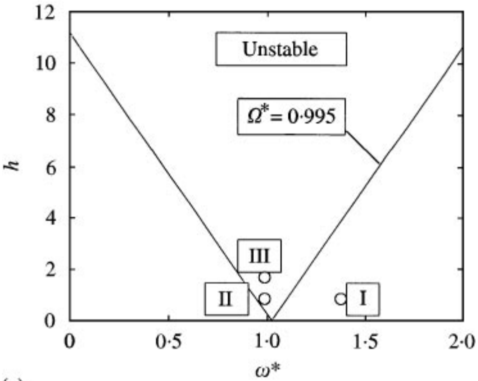


Figure 2. h–ω^* diagrams for $\Omega^* = 1$ and various values of k^2.

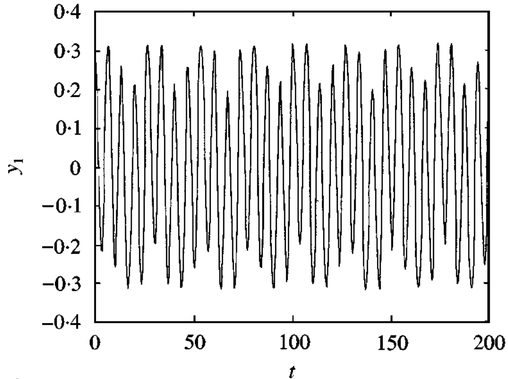
As seen, the boundaries depend on the initial amplitude, and on the excitation parameters: frequency and modulus of the Jacobi elliptic function. In Figure 1 the influence of Ω^* is shown for $k^2 = 0{\cdot}5$. It is concluded that the parameter Ω^* has an influence on the width of the unstable region: the higher the values of Ω^* the narrower the unstable district. It has also an effect on the position of the unstable region.

In Figure 2, the $h - \omega^*$ diagram is plotted for various values of k^2. It can be seen that the region of instability is a function of the modulus k^2 of the excitation function: the higher the value of the modulus the narrower the region.



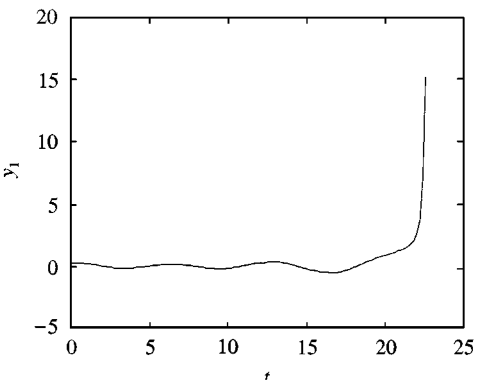
(a)

Figure 3(a). h–ω^* diagram for $\Omega^* = 0{\cdot}955$.



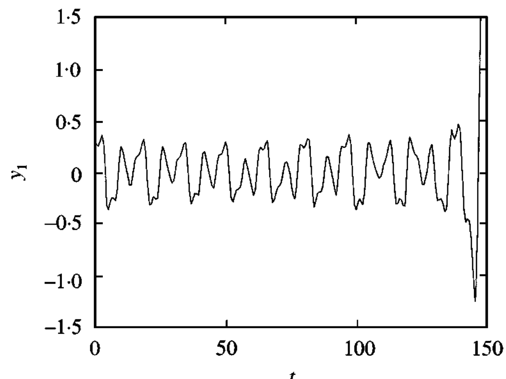
(b)

Figure 3(b). y_1–t diagram for point I.



(c)

Figure 3(c). y_1–t diagram for point II.



(d)

Figure 3(d). y_1–t diagram for point III.

3.1. EXAMPLE

To verify the accuracy of the analytical method and its result (equation (47)) a comparison is made with numerical solutions of equation (3). Consider the system where

$$\omega^2 = 1, \quad c_3^* = 1, \quad k^2 = 1/2.$$

For initial conditions $A = 0{\cdot}3$ and $\theta = \mathrm{K} = 1{\cdot}85407\ldots$, $\Omega^* = 0{\cdot}955$. Using the analytical solution (47) the stable and unstable regions are plotted in Figure 3(a). Three points are selected. Point 1 has the parameters $\Omega = \sqrt{2}$ and $h = 0{\cdot}5$. The co-ordinates of point II are $\Omega = 2$ and $h = 0{\cdot}5$. Point III has the same value of Ω and $h = 1{\cdot}3$. Applying the Runge–Kutta numerical procedure the solutions of equation (3) for these three sets of parameters are obtained. In Figure 3(b) the time-history diagram for point I is plotted. The motion is periodical and is in the stable region. In Figure 3(c) the displacement–time diagram for point II is plotted.

The curve has a tendency to diverge. The motion is in the unstable region. The time-history diagram for the parameters which correspond to point III is plotted in Figure 3(d). The motion is unstable. It has a tendency to increase. Discussing the results in Figure 3 it can be concluded that the regions of stability obtained analytically (Figure 3(a)) in the first approximation correspond to the real boundaries obtained numerically (Figures 3(b) and 3(c)) only for the case where the parameter $h \leqslant 1$. For $h > 1$ the analytical results are incorrect (Figure 3(d)). According to equation (4) it means that the analytical solving method is applicable only for $\varepsilon^* \leqslant \delta$.

4. EXACT SOLUTION

For equation (3) the exact solution can be obtained. Assume the solution in the form of the second order approximation as suggested in reference [20],

$$y_1 = A + B\,\mathrm{sn}(\Omega t, k^2), \tag{48}$$

where A and B are constants which have to be determined. This solution is in some sense a second order approximation. The time derivatives of equation (48) are

$$\dot{y}_1 = B\Omega\,\mathrm{cn}\,\mathrm{dn},$$

$$\ddot{y}_1 = -\,B\Omega^2\,\mathrm{sn}(1 + k^2 - 2k^2\,\mathrm{sn}^2),$$

where

$$\mathrm{sn} \equiv \mathrm{sn}(\Omega t, k^2), \quad \mathrm{cn} \equiv \mathrm{cn}(\Omega t, k^2), \quad \mathrm{dn} \equiv \mathrm{dn}(\Omega t, k^2).$$

Substituting equation (48) into equation (3) gives

$$-\,B\Omega^2\,\mathrm{sn}(1 + k^2 - 2k^2\,\mathrm{sn}^2) + \omega^2(A + B\,\mathrm{sn}) - \omega^2 h(A\,\mathrm{sn} + B\,\mathrm{sn}^2).$$

$$-\,c_3^*(A^3 + 3A^2B\,\mathrm{sn} + 3AB^2\,\mathrm{sn}^2 + B^3\,\mathrm{sn}^3) = 0. \tag{49}$$

Separating the terms with the same order of function sn one obtains

$$-\,\omega^2 + c_3^* A^2 = 0, \tag{50}$$

$$-\,B\Omega^2(1 + k^2) + \omega^2 B - \omega^2 hA - 3c_3^* A^2 B = 0, \tag{51}$$

$$\omega^2 h + 3c_3^* AB = 0, \tag{52}$$

$$-\,2k^2\Omega^2 + c_3^* B^2 = 0, \tag{53}$$

where

$$B = \frac{\omega h}{3\sqrt{c_3^*}}, \quad A = \sqrt{\frac{\omega^2}{c_3^*}},$$

$$k = \frac{h}{\sqrt{18 - h^2}}, \quad \Omega = \omega\sqrt{1 - \frac{h^2}{18}}. \tag{54}$$

It means that for the case where the excitation force is

$$F = F_0 \,\mathrm{sn}\left(\omega t\sqrt{1 - \frac{h^2}{18}}, \frac{h^2}{18 - h^2}\right), \tag{55}$$

the solution of equation (3) is

$$y_1 = \sqrt{\frac{\omega^2}{c_3^*} + \frac{\omega h}{3\sqrt{c_3^*}}}\,\mathrm{sn}\left(\omega t\sqrt{1 - \frac{h^2}{18}}, \frac{h^2}{18 - h^2}\right). \tag{56}$$

5. VIBRATIONS OF THE SYSTEM WITH LINEAR ABSORBER

As shown in section 3, depending on the excitation parameter Ω, the motion of the system (3) is stable or unstable. If the motion is unstable for a certain value of Ω one requires to turn it into a stable motion. The stabilization of the motion and the increasing of the region of stability is possible by adding a linear dynamic absorber to the aforementioned system (Figure 4). It is a mass–spring system. The

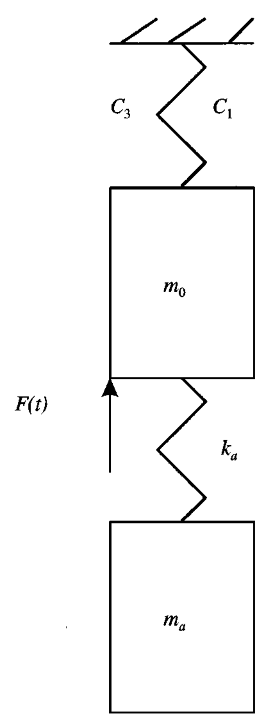


Figure 4. The system with linear absorber.

parameters of the absorber are the mass m_a and the rigidity of a spring k_a. For the special values of absorber parameters the motion of the system had to be periodical. The aim of the paper is to determine the values of the parameters of the absorber which stabilizes the motion of the basic system.

The differential equation of motion of the basic system with absorber is

$$\ddot{y}_1 + [\delta - \varepsilon F(t)]\, y_1 - c_3^* y_1^3 + m\alpha_a(y_1 - y_3) = 0, \tag{57}$$

$$\ddot{y}_3 + \alpha_a(y_3 - y_1) = 0, \tag{58}$$

where y_1 and y_3 are the deflection functions of the masses,

$$\alpha_a = \frac{k_a}{m_a}, \quad m = \frac{m_a}{m_0}, \tag{59}$$

with m_0 being the mass of the vibrating system. The initial conditions are

$$y_1(0) = y_{10}, \quad \dot{y}_1(0) = \dot{y}_{10}, \quad y_3(0) = y_{30}, \quad \dot{y}_3(0) = \dot{y}_{30}. \tag{60}$$

Eliminating y_3 from equation (57), gives

$$y_3 = \frac{1}{m\alpha_a}\{\ddot{y}_1 + [\delta - \varepsilon F(t)]\, y_1 - c_3^* y_1^3 + m\alpha_a y_1\}, \tag{61}$$

and substituting into equation (58) gives

$$y_1^{IV} + [\delta - \varepsilon F(t)]\, \ddot{y}_1 - \varepsilon \ddot{F} y_1 - 2\varepsilon \dot{y}_1 \dot{F} - 3c_3^* y_1(2\dot{y}_1^2 + y_1 \ddot{y}_1)$$
$$+ m\alpha_a \ddot{y}_1 + \alpha_a\{\ddot{y}_1 + [\delta - \varepsilon F(t)]\, y_1 - c_3^* y_1^3\} = 0. \tag{62}$$

According to equations (60) and (61) the initial conditions are

$$y_1(0) = y_{10}, \quad \dot{y}_1(0) = \dot{y}_{10},$$
$$\ddot{y}_1(0) = m\alpha_a(y_{30} - y_{10}) - [\delta - \varepsilon F(0)]\, y_{10} + c_3^* y_{10}^3,$$
$$\dddot{y}_1(0) \sim m\alpha_a(\dot{y}_{30} - \dot{y}_{10}) - [\delta - \varepsilon F(0)]\, \dot{y}_{10} + \varepsilon \dot{F}(0)\, y_{10} + 3c_3^* y_{10}^2 \dot{y}_{10}. \tag{63}$$

Now assume the excitation function (2). The sinus Jacobi elliptic function is shown as a series of trignometric function (see reference [19])

$$\operatorname{sn}(\Omega t, k^2) = \frac{2\pi}{kK} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{\pi K'(n+1)/2\mathrm{K}}}{1 - \mathrm{e}^{-\pi K'(2n+1)/\mathrm{K}}} \sin \frac{\pi(2n+1)\Omega t}{2\mathrm{K}}, \tag{64}$$

where $K = K(k^2) = K'(1 - k^2)$. Assuming only the first term and developing it into a Taylor series gives

$$\mathrm{sn}(\Omega t, k^2) \approx a(1 - b\cos 2\hat{\Omega} t), \tag{65}$$

where

$$\hat{\Omega} = \frac{\pi\Omega}{2K}, \quad a = \frac{\pi\sqrt{2}}{kK}\frac{e^{-\pi K'/2K}}{1 - e^{-\pi K'/K}}, \quad b = \frac{1}{2}. \tag{66}$$

The excitation function (2) transforms to

$$F(t) = F_0 a(1 - b\cos 2\hat{\Omega} t). \tag{67}$$

According to equation (67) equation (62) is

$$\begin{aligned} y_1^{IV} &+ [\delta + (m+1)\alpha_a - \varepsilon^* a(1 - b\cos 2\hat{\Omega} t)]\ddot{y}_1 \\ &+ y_1[\alpha_a\delta - \varepsilon^* a\alpha_a + ab\varepsilon^*(\alpha_a - 4\hat{\Omega}^2)\cos 2\hat{\Omega} t] \\ &- 4\varepsilon^* ab\Omega_1\dot{y}_1\sin 2\Omega_1 t - 3c_3^* y_1(2\dot{y}_1^2 + y_1\ddot{y}_1) - \alpha_a c_3^* y_1^3 \\ &= 0, \end{aligned} \tag{68}$$

and the initial conditions (63) are

$$\begin{aligned} y_1(0) &= y_{10}, \quad \dot{y}_1(0) = \dot{y}_{10}, \\ \ddot{y}_1(0) &= m\alpha_a(y_{30} - y_{10}) - [\delta - \varepsilon^* a(1 - b)]y_{10} + c_3^* y_{10}^3, \\ \dddot{y}_1(0) &= m\alpha_a(\dot{y}_{30} - \dot{y}_{10}) - [\delta - \varepsilon^* a(1 - b)]\dot{y}_{10} + 3c_3^* y_{10}^2\dot{y}_{10}. \end{aligned} \tag{69}$$

where

$$\varepsilon^* = \varepsilon F_0. \tag{70}$$

Now assume the solution of equation (68) as

$$y_1 = A\cos\hat{\Omega} t. \tag{71}$$

If equation (71) is substituted into equation (68), one obtains

$$\begin{aligned} \hat{\Omega}^4 &- [\delta + (m+1)\alpha_a - \varepsilon^* a + \varepsilon^* ab(2\cos^2\hat{\Omega} t - 1]\hat{\Omega}^2 \\ &+ [-\alpha_a a\varepsilon^* + ab\varepsilon^*(\alpha_a - 4\hat{\Omega}^2)\, 2\cos^2\hat{\Omega} t - 1)] \\ &+ 8ab\varepsilon^*\hat{\Omega}^2(1 - \cos^2\hat{\Omega} t) - 3c_3^* A^2\hat{\Omega}^2(2 - 3\cos^2\hat{\Omega} t) \\ &- \alpha_a c_3^* A^2\cos^2\hat{\Omega} t \\ &= 0. \end{aligned} \tag{72}$$

Separating the terms with cos and $\cos^3$ yields a system of two algebraic equations

$$\hat{\Omega}^4 - [\delta + (m+1)\alpha_a - \varepsilon^* a(1+b)]\hat{\Omega}^2 - \varepsilon^* a[\alpha_a + b(\alpha_a - 4\hat{\Omega}^2)] + 8\varepsilon^* ab\hat{\Omega}^2 - 6c_3^* A^2\hat{\Omega}^2 = 0 \quad (73)$$

and

$$-18\hat{\Omega}^2\varepsilon^* ab + 2\varepsilon^* ab(\alpha_a - 4\hat{\Omega}^2) - \alpha_a c_3^* A^2 + 9c_3^* A^2\hat{\Omega}^2 = 0. \quad (74)$$

From equation (74), one obtains the parameter of the absorber

$$\alpha_a = 9\hat{\Omega}^2. \quad (75)$$

From equation (73) the parameter of the mass ratio is

$$m = \frac{1}{9\hat{\Omega}^2}(8\delta - 8\hat{\Omega}^2 - 7a\varepsilon^* - 6c_3^* A^2). \quad (76)$$

The parameter α_a of the absorber depends on the frequency of parametric excitation. The mass ratio is a function of the excitation coefficients and the parameters of the primary system.

The influence of the linear absorber on the vibrations of the system is shown in Figure 5. In Figure 5(a), the $y_1 - t$ diagram for $\hat{\Omega} = 0{\cdot}5$, $\varepsilon^* = 0{\cdot}5$, $\delta = 0{\cdot}54$, $a = 0{\cdot}21727$, $c_3^* = 1$ is plotted. The motion is unstable. In Figure 5(b), the $y_{1N} - t$ and $y_{1A} - t$ diagrams of the system with the linear absorber whose parameters are $\alpha_a = 2{\cdot}25$ and $m = 0{\cdot}3$ are plotted. Equations (57) and (58) for excitation (2) are

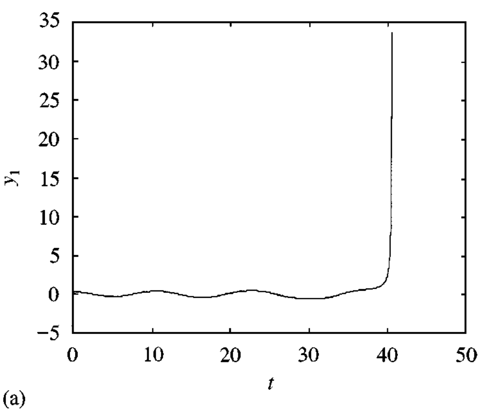


Figure 5(a) y_1–t diagram for the basic system without absorber.

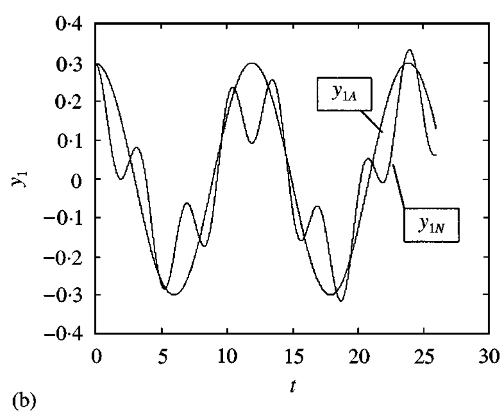


Figure 5(b). Displacement–time diagrams for the system with absorber obtained analytically (y_{1A}) and numerically (y_{1N}).

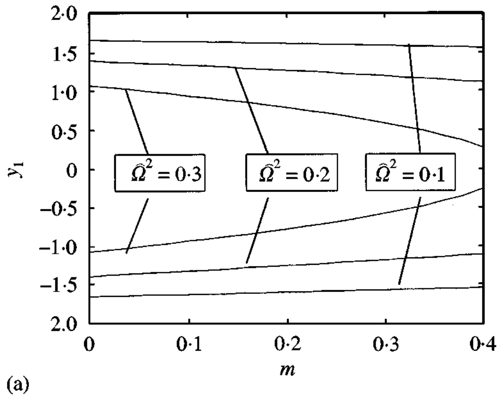


Figure 6(a). A–m diagrams of the negative non-linear system for various excitation parameters.

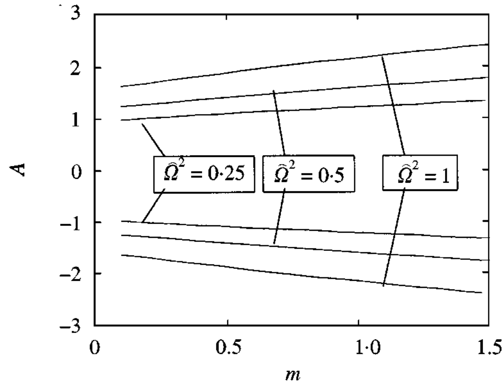


Figure 6(b). A–m diagrams of the positive non-linear system for various excitation parameters.

solved numerically applying the Runge–Kutta method. The numerically obtained solution y_{1N} is compared with the analytical solution (71) y_{1A}. It can be seen that the analytical solution is an averaged value of the real numerical solution. The motion of the system is stabilized with the linear absorber.

The amplitude-mass ratio diagrams for various values of excitation frequency $\hat{\Omega}$ are plotted in Figure 6. The constants of the system are: $\varepsilon^* = 0{\cdot}5$, $\delta = 0{\cdot}54$, $a = 0{\cdot}21727$. In Figure 6(a) $c_3^* = 1$, and in Figure 6(b) $c_3^* = -1$. It can be concluded that for the positive non-linearity the amplitude of vibrations has a tendency to increase by increasing the mass ratio parameter: the larger the excitation parameter $\hat{\Omega}$ the faster the increase. For the negative non-linearity the amplitude of vibrations has a tendency to decrease with increase of the mass ratio parameter. The decrease is faster for higher values of excitation parameter $\hat{\Omega}$.

6. CONCLUSION

This paper analyses the parametrically excited vibrations in the strong non-linear system. The following is concluded:

1. In the strong non-linear system where the excitation is periodical, the vibrations are described with the second order non-linear differential equation with variable parameters. For the case where the excitation is given with a function which is of Jacobi elliptic type the vibrations can be determined by applying the approximate analytical elliptic-Krylov–Bogolubov method. The solution of the differential equation is bounded or unbounded depending on the parameters of the system. The region of instability depends not only on the amplitude of excitation h, the ratio between the frequency of the system and of the excitation ω^* (as it is the case for linear systems), but also on the modulus of the excitation force k^2 and the initial amplitude A.
2. For some special excitation, the particular solution of vibration can be obtained.

3. The linear dynamic absorber which is attached to the strong non-linear parameterically excited system can stabilize the motion and provide periodical motion. The parameters of the mass-spring system are calculated. They depend on the parameter of excitation and the parameters of the original system.

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APPENDIX A

$$\int_0^{4K} \frac{b}{i} \frac{\mathrm{sn}^2\,\psi}{(1-k^2\,\mathrm{sn}^4\,\psi)}\,\mathrm{d}\psi$$

$$= -\Bigg[-4\,\mathrm{cn}^3\,\theta\,\mathrm{dn}\,\theta\,\mathrm{sn}^3\,\theta \int_0^{4K} (1-k^2\,\mathrm{sn}^4\,\psi) k^4\,\mathrm{sn}^6\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{d}\psi$$

$$-\,4k^2\,\mathrm{cn}\,\theta\,\mathrm{dn}^3\,\theta\,\mathrm{sn}^3\,\theta \int_0^{4K} (1-k^2\,\mathrm{sn}^4)\,\mathrm{sn}^6\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi$$

$$+\,4\,\mathrm{cn}\,\theta\,\mathrm{dn}^3\,\theta\,\mathrm{sn}^5\,\theta \int_0^{4K} (1+k^2\,\mathrm{sn}^4\,\psi)\,k^4\,\mathrm{sn}^8\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi$$

$$+4k^2\,\mathrm{sn}^3\,\theta\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta \int_0^{4K} \mathrm{sn}^4\,\psi(1+k^2\,\mathrm{sn}^4\,\psi)\,\mathrm{cn}^2\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi$$

$$-4\,\mathrm{sn}^7\,\theta\,\mathrm{dn}\,\theta\,\mathrm{cn}\,\theta \int_0^{4K} (1+k^2\,\mathrm{sn}^4\,\psi)\,k^6\,\mathrm{sn}^8\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi$$

$$+4k^6\,\mathrm{cn}^3\,\theta\,\mathrm{dn}\,\theta\,\mathrm{sn}^5\,\theta \int_0^{4K} (1+k^2\,\mathrm{sn}^4\,\psi)\,\mathrm{sn}^8\,\psi\,\mathrm{cn}^2\,\psi$$

$$+6k^4\,\mathrm{sn}^5\,\theta\,\mathrm{cn}^3\,\theta\,\mathrm{dn}\,\theta\,\mathrm{sn}\,\theta \int_0^{4K} (1+k^2\,\mathrm{sn}^4\,\psi)\,k^2\,\mathrm{sn}^8\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{d}\psi$$

$$+6k^4\,\mathrm{sn}^5\,\theta\,\mathrm{cn}\,\theta\,\mathrm{dn}^3\,\theta\,\mathrm{sn}\,\theta \int_0^{4K} (1+k^2\,\mathrm{sn}^4\,\psi)\,\mathrm{sn}^8\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi$$

$$-6k^4\,\mathrm{sn}^7\,\theta\,\mathrm{cn}\,\theta\,\mathrm{dn}^3\,\theta \int_0^{4K} (1+k^2\,\mathrm{sn}^4\,\psi)\,k^2\,\mathrm{sn}^{10}\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi$$

$$-6k^4\,\mathrm{sn}^5\,\theta\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta \int_0^{4K} (1+k^2\,\mathrm{sn}^4\,\psi)\,\mathrm{sn}^6\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi$$

$$+6k^4\,\mathrm{sn}^9\,\theta\,\mathrm{dn}\,\theta\,\mathrm{cn}\,\theta \int_0^{4K} (1+k^2\,\mathrm{sn}^4\,\psi)\,k^4\,\mathrm{sn}^{10}\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{dn}^2\,\psi\,\mathrm{d}\psi$$

$$-6k^4\,\mathrm{sn}^7\,\theta\,\mathrm{cn}^3\,\theta\,\mathrm{dn}\,\theta \int_0^{4K} (1+k^2\,\mathrm{sn}^4\,\psi)\,k^4\,\mathrm{sn}^{10}\,\psi\,\mathrm{cn}^2\,\psi\,\mathrm{d}\psi$$

$$-4k^6 \operatorname{sn}^7\theta \operatorname{cn}^3\theta \operatorname{dn}\theta \int_0^{4K} (1+k^2\operatorname{sn}^4\psi)\, k^2 \operatorname{sn}^{10}\psi \operatorname{cn}^2\psi$$

$$-4k^6 \operatorname{sn}^7\theta \operatorname{cn}\theta \operatorname{dn}^3\theta \int_0^{4K} (1+k^2\operatorname{sn}^4\psi) \operatorname{sn}^{10}\psi \operatorname{dn}^2\psi$$

$$+4k^6 \operatorname{sn}^9\theta \operatorname{cn}\theta \operatorname{dn}^3\theta \int_0^{4K} (1+k^2\operatorname{sn}^4\psi)\, k^2 \operatorname{sn}^{12}\psi \operatorname{dn}^2\psi$$

$$+4k^6 \operatorname{sn}^7\theta \operatorname{cn}\theta \operatorname{dn}\theta \int_0^{4K} \operatorname{sn}^8\psi (1+k^2\operatorname{sn}^4\psi) \operatorname{cn}^2\psi \operatorname{dn}^2\psi$$

$$-4k^6 \operatorname{sn}^{11}\theta \operatorname{dn}\theta \operatorname{cn}\theta \int_0^{4K} (1+k^2\operatorname{sn}^4\psi)\, k^4 \operatorname{sn}^{12}\psi \operatorname{cn}^2\psi \operatorname{dn}^2\psi$$

$$+4k^6 \operatorname{sn}^9\theta \operatorname{cn}^3\theta \operatorname{dn}\theta \int_0^{4K} (1+k^2\operatorname{sn}^4\psi)\, k^4 \operatorname{sn}^{12}\psi \operatorname{cn}^2\psi$$

$$-k^2 \operatorname{sn}^5\theta \operatorname{cn}^3\theta \operatorname{dn}\theta \int_0^{4K} (1+k^2\operatorname{sn}^4\psi)\, k^2 \operatorname{sn}^4\psi \operatorname{cn}^6\psi \operatorname{dn}^4\psi \, d\psi$$

$$-k^2 \operatorname{sn}^5\theta \operatorname{cn}\theta \operatorname{dn}^8\theta \int_0^{4K} (1+k^2\operatorname{sn}^4\psi) \operatorname{sn}^4\psi \operatorname{dn}^6\psi \operatorname{cn}^4\psi \, d\psi$$

$$+k^2 \operatorname{sn}^7\theta \operatorname{cn}\theta \operatorname{dn}^3\theta \int_0^{4K} (1+k^2\operatorname{sn}^4\psi)\, k^2 \operatorname{sn}^6\psi \operatorname{dn}^6\psi \operatorname{cn}^4\psi \, d\psi$$

$$+k^2 \operatorname{sn}^5\theta \operatorname{cn}\theta \operatorname{dn}\theta \operatorname{sn}^2\psi \int_0^{4K} (1+k^2\operatorname{sn}^4\psi) \operatorname{cn}^6\psi \operatorname{dn}^6\psi \, d\psi$$

$$-k^2 \operatorname{sn}^9\theta \operatorname{dn}\theta \operatorname{cn}\theta \int_0^{4K} (1+k^2\operatorname{sn}^4\psi)\, k^4 \operatorname{sn}^6\psi \operatorname{cn}^6\psi \operatorname{dn}^6\psi \, d\psi$$

$$+\operatorname{cn}^3\theta \operatorname{dn}\theta \operatorname{sn}^7\theta \int_0^{4K} \operatorname{dn}^4\psi (1+k^2\operatorname{sn}^4\psi)\, k^6 \operatorname{sn}^6\psi \operatorname{cn}^5\psi \, d\psi$$

$$+\operatorname{cn}^3\theta \operatorname{dn}\theta \operatorname{sn}^9\theta \int_0^{4K} (1+k^2\operatorname{sn}^4\psi)\, k^{18} \operatorname{sn}^{12}\psi \operatorname{cn}^2\psi \, d\psi$$

$$+\operatorname{cn}\theta \operatorname{dn}^3\theta \operatorname{sn}^9\theta k^8 \int_0^{4K} (1+k^2\operatorname{sn}^4\psi) \operatorname{sn}^{12}\psi \operatorname{dn}^2\psi \, d\psi$$

$$-\operatorname{cn}\theta\operatorname{dn}^3\theta\operatorname{sn}^{11}\theta\int_0^{4K}(1+k^2\operatorname{sn}^4\psi)\,k^{10}\operatorname{sn}^{14}\psi\operatorname{dn}^2\psi\,d\psi$$

$$-\operatorname{sn}^9\theta\operatorname{cn}\theta\operatorname{dn}\theta\,k^8\int_0^{4K}\operatorname{sn}^{10}\psi(1+k^2\operatorname{sn}^4\psi)\operatorname{cn}^2\psi\operatorname{dn}^2\psi\,d\psi$$

$$+\operatorname{sn}^{13}\theta\operatorname{dn}\theta\operatorname{cn}\theta\int_0^{4K}(1+k^2\operatorname{sn}^4\psi)\,k^{12}\operatorname{sn}^{14}\psi\operatorname{cn}^2\psi\operatorname{dn}^2\psi\,d\psi$$

$$-\operatorname{cn}^3\theta\operatorname{dn}\theta\operatorname{sn}^{11}\theta\int_0^{4K}(1+k^2\operatorname{sn}^4\psi)\,k^{12}\operatorname{sn}^{14}\psi\operatorname{cn}^2\psi\,d\psi$$

$$-\operatorname{cn}^7\theta\operatorname{dn}^5\theta\operatorname{sn}\theta\int_0^{4K}(1+k^2\operatorname{sn}^4\psi)\,k^4\operatorname{sn}^8\psi\operatorname{cn}^2\psi\,d\psi$$

$$-k^2\operatorname{cn}^5\theta\operatorname{dn}^7\theta\operatorname{sn}\theta\int_0^{4K}(1+k^2\operatorname{sn}^4\psi)\operatorname{sn}^8\psi\operatorname{dn}^2\psi\,d\psi$$

$$+\operatorname{cn}^5\theta\operatorname{dn}^7\theta\operatorname{sn}^3\theta\int_0^{4K}(1+k^2\operatorname{sn}^4\psi)\,k^4\operatorname{sn}^{10}\psi\operatorname{dn}^2\psi\,d\psi$$

$$+\operatorname{sn}\theta\operatorname{cn}^5\theta\operatorname{dn}^5\theta k^2\int_0^{4K}\operatorname{sn}^6\psi(1+k^2\operatorname{sn}^4\psi)\operatorname{cn}^2\psi\operatorname{dn}^2\psi\,d\psi$$

$$-\operatorname{sn}^5\theta\operatorname{dn}^5\theta\operatorname{cn}^5\theta\int_0^{4K}(1+k^2\operatorname{sn}^4\psi)\,k^6\operatorname{sn}^{10}\psi\operatorname{cn}^2\psi\operatorname{dn}^2\psi\,d\psi$$

$$+\operatorname{cn}^7\theta\operatorname{dn}^5\theta\operatorname{sn}^3\theta\int_0^{4K}(1+k^2\operatorname{sn}^4\psi)\,k^6\operatorname{sn}^{10}\psi\operatorname{cn}^2\psi\,d\psi$$

$$-6\operatorname{cn}^3\theta\operatorname{dn}\theta\operatorname{sn}\theta\int_0^{4K}(1+k^2\operatorname{sn}^4\psi)\,k^4\operatorname{sn}^6\psi\operatorname{cn}^4\psi\,d\psi$$

$$-6k^2\operatorname{cn}\theta\operatorname{dn}^3\theta\operatorname{sn}\theta\int_0^{4K}\operatorname{cn}^2\psi(1+k^2\operatorname{sn}^4\psi)\operatorname{sn}^6\psi\operatorname{dn}^2\psi\,d\psi$$

$$+6\operatorname{cn}\theta\operatorname{dn}^3\theta\operatorname{sn}^3\theta\int_0^{4K}\operatorname{cn}^2\psi(1+k^2\operatorname{sn}^4\psi)\,k^4\operatorname{sn}^8\psi\operatorname{dn}^2\psi\,d\psi$$

$$+6k^2\operatorname{sn}\theta\operatorname{cn}\theta\operatorname{dn}\theta\int_0^{4K}\operatorname{sn}^2\psi(1+k^2\operatorname{sn}^4\psi)\operatorname{cn}^4\psi\operatorname{dn}^2\psi\,d\psi$$

$$-6\mathrm{sn}^5\theta\,\mathrm{dn}\,\theta\,\mathrm{cn}\,\theta\int_0^{4K}(1+k^2\,\mathrm{sn}^4\psi)\,k^6\,\mathrm{sn}^8\psi\,\mathrm{cn}^4\psi\,\mathrm{dn}^2\psi\,\mathrm{d}\psi$$

$$+6\mathrm{cn}^3\theta\,\mathrm{dn}\,\theta\,\mathrm{sn}^3\theta\int_0^{4K}(1+k^2\,\mathrm{sn}^4\psi)\,k^6\,\mathrm{sn}^8\psi\,\mathrm{cn}^4\psi\,\mathrm{d}\psi$$

$$+\mathrm{cn}^5\theta\,\mathrm{dn}^3\theta\,\mathrm{sn}^3\theta\int_0^{4K}\mathrm{dn}^2\psi(1+k^2\,\mathrm{sn}^4\psi)\,k^2\,\mathrm{sn}^4\psi\,\mathrm{cn}^2\psi\,\mathrm{d}\psi$$

$$+\mathrm{cn}^3\theta\,\mathrm{dn}^5\theta\,\mathrm{sn}^3\theta\int_0^{4K}(1+k^2\,\mathrm{sn}^4\psi)\,\mathrm{sn}^4\psi\,\mathrm{dn}^4\psi\,\mathrm{d}\psi$$

$$-\mathrm{cn}^3\theta\,\mathrm{dn}^5\theta\,\mathrm{sn}^5\theta\int_0^{4K}(1+k^2\,\mathrm{sn}^4\psi)\,k^2\,\mathrm{sn}^6\psi\,\mathrm{dn}^4\psi\,\mathrm{d}\psi$$

$$-\mathrm{sn}^3\theta\,\mathrm{cn}^3\theta\,\mathrm{dn}^3\theta\int_0^{4K}\mathrm{sn}^2\psi(1+k^2\,\mathrm{sn}^4\psi)\,\mathrm{cn}^2\psi\,\mathrm{dn}^4\psi\,\mathrm{d}\psi$$

$$+\mathrm{sn}^7\theta\,\mathrm{dn}^3\theta\,\mathrm{cn}^3\theta\int_0^{4K}(1+k^2\,\mathrm{sn}^4\psi)\,k^4\,\mathrm{sn}^6\psi\,\mathrm{cn}^2\psi\,\mathrm{dn}^4\psi\,\mathrm{d}\psi$$

$$-\mathrm{cn}^5\theta\,\mathrm{dn}^3\theta\,\mathrm{sn}^5\theta\int_0^{4K}\mathrm{dn}^2\psi(1+k^2\,\mathrm{sn}^4\psi)\,k^4\,\mathrm{sn}^6\psi\,\mathrm{cn}^2\psi\,\mathrm{d}\psi$$

$$-\mathrm{cn}^3\theta\,\mathrm{dn}\,\theta\,\mathrm{sn}\,\theta\int_0^{4K}(1+k^2\,\mathrm{sn}^4\psi)\,k^2\,\mathrm{sn}^4\psi\,\mathrm{cn}^2\psi\,\mathrm{d}\psi$$

$$-\mathrm{cn}\,\theta\,\mathrm{dn}^3\theta\,\mathrm{sn}\,\theta\int_0^{4K}(1+k^2\,\mathrm{sn}^4\psi)\,\mathrm{sn}^4\psi\,\mathrm{dn}^2\psi\,\mathrm{d}\psi$$

$$+\mathrm{cn}\,\theta\,\mathrm{dn}^3\theta\,\mathrm{sn}^3\theta\int_0^{4K}(1+k^2\,\mathrm{sn}^4\psi)\,k^2\,\mathrm{sn}^6\psi\,\mathrm{dn}^2\psi\,\mathrm{d}\psi$$

$$+\mathrm{sn}\,\theta\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta\int_0^{4K}\mathrm{sn}^2\psi(1+k^2\,\mathrm{sn}^4\psi)\,\mathrm{cn}^2\psi\,\mathrm{dn}^2\psi\,\mathrm{d}\psi$$

$$-\mathrm{sn}^5\theta\,\mathrm{dn}\,\theta\,\mathrm{cn}\,\theta\int_0^{4K}(1+k^2\,\mathrm{sn}^4\psi)\,k^4\,\mathrm{sn}^6\psi\,\mathrm{cn}^2\psi\,\mathrm{d}^2\psi\,\mathrm{d}\psi$$

$$+\mathrm{cn}^3\theta\,\mathrm{dn}\,\theta\,\mathrm{sn}^3\theta\int_0^{4K}(1+k^2\,\mathrm{sn}^4\psi)\,k^4\,\mathrm{sn}^6\psi\,\mathrm{cn}^2\psi\,\mathrm{d}\psi\Bigg]$$

$$+4k^2 \operatorname{cn}^5\theta \operatorname{dn}^5\theta \operatorname{sn}\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) \operatorname{sn}^6\psi \operatorname{cn}^2\psi \operatorname{dn}^2\psi \, d\psi$$

$$+4\operatorname{cn}^3\theta \operatorname{dn}^5\theta \operatorname{sn}^5\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) k^4 \operatorname{sn}^8\psi \operatorname{cn}^2\psi \operatorname{dn}^4\psi \, d\psi$$

$$-4\operatorname{cn}^5\theta \operatorname{dn}^5\theta \operatorname{sn}^5\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) k^6 \operatorname{sn}^{10}\psi \operatorname{cn}^2\psi \operatorname{dn}^2\psi \, d\psi$$

$$-4\operatorname{cn}^5\theta \operatorname{dn}^3\theta \operatorname{sn}^3\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) k^4 \operatorname{sn}^6\psi \operatorname{cn}^4\psi \operatorname{dn}^2\psi \, d\psi$$

$$+4\operatorname{cn}^5\theta \operatorname{dn}^3\theta \operatorname{sn}^5\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) k^6 \operatorname{sn}^8\psi \operatorname{cn}^4\psi \operatorname{dn}^2\psi \, d\psi$$

$$-4k^2 \operatorname{cn}^3\theta \operatorname{dn}^5\theta \operatorname{sn}^3\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) \operatorname{sn}^6\psi \operatorname{cn}^2\psi \operatorname{dn}^4\psi \, d\psi$$

$$+4k^2 \operatorname{cn}^3\theta \operatorname{dn}^3\theta \operatorname{sn}^3\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) \operatorname{sn}^4\psi \operatorname{cn}^4\psi \operatorname{dn}^4\psi \, d\psi$$

$$+4\operatorname{cn}\theta \operatorname{dn}^5\theta \operatorname{sn}^7\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) k^4 \operatorname{sn}^6\psi \operatorname{cn}^4\psi \operatorname{dn}^6\psi \, d\psi$$

$$-4\operatorname{cn}^3\theta \operatorname{dn}^3\theta \operatorname{sn}^7\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) k^6 \operatorname{sn}^8\psi \operatorname{cn}^4\psi \operatorname{dn}^4\psi \, d\psi$$

$$-4\operatorname{cn}^3\theta \operatorname{dn}\theta \operatorname{sn}^5\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) k^4 \operatorname{sn}^4\psi \operatorname{cn}^6\psi \operatorname{dn}^4\psi \, d\psi$$

$$+4\operatorname{cn}^3\theta \operatorname{dn}\theta \operatorname{sn}^7\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) k^6 \operatorname{sn}^6\psi \operatorname{cn}^6\psi \operatorname{dn}^4\psi \, d\psi$$

$$-4k^2 \operatorname{cn}\theta \operatorname{sn}^5\theta \operatorname{dn}^3\theta \int_0^{4K} (1+k^2 \operatorname{sn}^4\psi) \operatorname{sn}^4\psi \operatorname{cn}^4\psi \operatorname{dn}^6\psi \, d\psi, \tag{77}$$

$$\int_0^{4K} \frac{b}{i} \frac{\operatorname{sn}\psi \operatorname{cn}\psi \operatorname{dn}\psi}{(1-k^2 \operatorname{sn}^4\psi)} d\psi$$

$$= \operatorname{cn}^2\theta \operatorname{dn}^2\theta \int_0^{4K} \operatorname{sn}^2\psi \operatorname{cn}^2\psi \operatorname{dn}^2\psi (1+k^2 \operatorname{sn}^4\psi) \, d\psi$$

$$+\operatorname{sn}^4\theta\operatorname{dn}^2\theta\int_0^{4K}k^2\operatorname{sn}^4\psi\operatorname{cn}^2\psi\operatorname{dn}^4\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$-k^4\operatorname{cn}^2\theta\operatorname{dn}^2\theta\operatorname{sn}^4\theta\int_0^{4K}\operatorname{sn}^6\psi\operatorname{cn}^2\psi\operatorname{dn}^2\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$-k^2\operatorname{sn}^2\theta\operatorname{cn}^2\theta\int_0^{4K}\operatorname{sn}^2\psi\operatorname{cn}^4\psi\operatorname{dn}^2\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$+k^4\operatorname{sn}^4\theta\operatorname{cn}^2\theta\int_0^{4K}\operatorname{sn}^4\psi\operatorname{cn}^4\psi\operatorname{dn}^2\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$-\operatorname{sn}^2\theta\operatorname{dn}^2\theta\int_0^{4K}\operatorname{sn}^2\psi\operatorname{cn}^2\psi\operatorname{dn}^4\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$+4k^4\operatorname{sn}^2\theta\operatorname{cn}^2\theta\operatorname{dn}^2\theta\int_0^{4K}\operatorname{sn}^4\psi\operatorname{cn}^2\psi\operatorname{dn}^2\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$+4k^4\operatorname{sn}^6\theta\operatorname{dn}^2\theta\int_0^{4K}\operatorname{sn}^6\psi\operatorname{cn}^2\psi\operatorname{dn}^4\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$-4k^6\operatorname{sn}^6\theta\operatorname{cn}^2\theta\operatorname{dn}^2\theta\int_0^{4K}\operatorname{sn}^8\psi\operatorname{cn}^2\psi\operatorname{dn}^2\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$-4k^4\operatorname{sn}^4\theta\operatorname{cn}^2\theta\int_0^{4K}\operatorname{sn}^4\psi\operatorname{cn}^2\psi\operatorname{dn}^2\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$+4k^6\operatorname{sn}^6\theta\operatorname{cn}^2\theta\int_0^{4K}\operatorname{sn}^6\psi\operatorname{cn}^4\psi\operatorname{dn}^2\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$-4k^2\operatorname{sn}^4\theta\operatorname{dn}^2\theta\int_0^{4K}\operatorname{sn}^4\psi\operatorname{cn}^2\psi\operatorname{dn}^4\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$-6k^4\operatorname{sn}^4\theta\operatorname{cn}^2\theta\operatorname{dn}^2\theta\int_0^{4K}\operatorname{sn}^{10}\psi\operatorname{cn}^2\psi\operatorname{dn}^2\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$-6k^6\operatorname{sn}^8\theta\operatorname{dn}^2\theta\int_0^{4K}\operatorname{sn}^8\psi\operatorname{cn}^2\psi\operatorname{dn}^4\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$+6k^8\operatorname{sn}^8\theta\operatorname{cn}^2\theta\operatorname{dn}^2\theta\int_0^{4K}\operatorname{sn}^{10}\psi\operatorname{cn}^2\psi\operatorname{dn}^2\psi(1+k^2\operatorname{sn}^4\psi)\,d\psi$$

$$+ 6k^6 \operatorname{sn}^6\theta \operatorname{cn}^2\theta \int_0^{4K} \operatorname{sn}^6\psi \operatorname{cn}^4\psi \operatorname{dn}^2\psi (1 + k^2 \operatorname{sn}^4\psi)\,d\psi$$

$$- 6k^8 \operatorname{sn}^8\theta \operatorname{cn}^2\theta \int_0^{4K} \operatorname{sn}^8\psi \operatorname{cn}^4\psi \operatorname{dn}^2\psi (1 + k^2 \operatorname{sn}^4\psi)\,d\psi$$

$$+ 6k^4 \operatorname{sn}^6\theta \operatorname{dn}^2\theta \int_0^{4K} \operatorname{sn}^6\psi \operatorname{cn}^2\psi \operatorname{dn}^4\psi (1 + k^2 \operatorname{sn}^4\psi)\,d\psi$$

$$+ 4k^4 \operatorname{cn}^6\theta \operatorname{dn}^4\theta \operatorname{sn}^2\theta \int_0^{4K} (1 + k^2 \operatorname{sn}^4\psi)\, k^2 \operatorname{sn}^6\psi \operatorname{cn}^4\psi \operatorname{dn}^2\psi\,d\psi$$

$$+ 4k^2 \operatorname{cn}^4\theta \operatorname{dn}^6\theta \operatorname{sn}^2\theta \int_0^{4K} (1 + k^2 \operatorname{sn}^4\psi) \operatorname{sn}^6\psi \operatorname{dn}^4\psi \operatorname{cn}^2\psi\,d\psi$$

$$- 4\operatorname{cn}^4\theta \operatorname{dn}^6\theta \operatorname{sn}^4\theta \int_0^{4K} (1 + k^2 \operatorname{sn}^4\psi)\, k^4 \operatorname{sn}^8\psi \operatorname{cn}^2\psi \operatorname{dn}^6\psi\,d\psi$$

$$- 4k^2 \operatorname{cn}^4\theta \operatorname{dn}^4\theta \operatorname{sn}^2\theta \int_0^{4K} (1 + k^2 \operatorname{sn}^4\psi) \operatorname{sn}^4\psi \operatorname{cn}^4\psi \operatorname{dn}^4\psi\,d\psi$$

$$+ 4k^2 \operatorname{cn}^4\theta \operatorname{dn}^4\theta \operatorname{sn}^6\theta \int_0^{4K} (1 + k^2 \operatorname{sn}^4\psi)\, k^4 \operatorname{sn}^8\psi \operatorname{cn}^4\psi \operatorname{dn}^4\psi\,d\psi$$

$$- 4k^2 \operatorname{cn}^6\theta \operatorname{dn}^4\theta \operatorname{sn}^4\theta \int_0^{4K} (1 + k^2 \operatorname{sn}^4\psi)\, k^4 \operatorname{sn}^8\psi \operatorname{cn}^4\psi \operatorname{dn}^2\psi\,d\psi$$

$$+ 4k^4 \operatorname{cn}^4\theta \operatorname{dn}^2\theta \operatorname{sn}^4\theta (1 + k^2 \operatorname{sn}^4\psi) \operatorname{sn}^4\psi \operatorname{cn}^6\psi \operatorname{dn}^4\psi\,d\psi$$

$$+ 4k^2 \operatorname{cn}^2\theta \operatorname{dn}^4\theta \operatorname{sn}^4\theta (1 + k^2 \operatorname{sn}^4\psi) \operatorname{sn}^4\psi \operatorname{cn}^4\psi \operatorname{dn}^6\psi\,d\psi$$

$$- 4k^4 \operatorname{cn}^2\theta \operatorname{dn}^4\theta \operatorname{sn}^6\theta \int_0^{4K} (1 + k^2 \operatorname{sn}^4\psi) \operatorname{sn}^6\psi \operatorname{cn}^4\psi \operatorname{dn}^8\psi\,d\psi$$

$$- 4k^2 \operatorname{sn}^4\theta \operatorname{cn}^2\theta \operatorname{dn}^2\theta \int_0^{4K} (1 + k^2 \operatorname{sn}^4\psi) \operatorname{sn}^2\psi \operatorname{cn}^6\psi \operatorname{dn}^6\psi\,d\psi$$

$$+ 4\operatorname{cn}^2\theta \operatorname{dn}^2\theta \operatorname{sn}^8\theta \int_0^{4K} (1 + k^2 \operatorname{sn}^4\psi)\, k^6 \operatorname{sn}^6\psi \operatorname{cn}^6\psi \operatorname{dn}^6\psi\,d\psi$$

$$- 4\operatorname{cn}^4\theta \operatorname{dn}^2\theta \operatorname{sn}^6\theta \int_0^{4K} (1 + k^2 \operatorname{sn}^4\psi)\, k^6 \operatorname{sn}^6\psi \operatorname{cn}^6\psi \operatorname{dn}^4\psi\,d\psi. \qquad (78)$$

Before the averaging it is very convenient to transform all the elliptic functions to sinus elliptic function

$$\mathrm{cn}^2\,\mathrm{dn}^2 = (1-\mathrm{sn}^2)(1-k^2\,\mathrm{sn}^2) = 1-(k^2+1)\,\mathrm{sn}^2+\mathrm{sn}^4 k^2.$$

$$\mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{dn}^2 = \mathrm{sn}^2-(k^2+1)\,\mathrm{sn}^4+\mathrm{sn}^6 k^2,$$

$$\mathrm{sn}^4\,\mathrm{cn}^2\,\mathrm{dn}^2 = \mathrm{sn}^4-(k^2+1)\,\mathrm{sn}^6+\mathrm{sn}^8 k^2,$$

$$\mathrm{sn}^6\,\mathrm{cn}^2\,\mathrm{dn}^2 = \mathrm{sn}^6-(k^2+1)\,\mathrm{sn}^8+\mathrm{sn}^{10} k^2,$$

$$\mathrm{sn}^8\,\mathrm{cn}^2\,\mathrm{dn}^2 = \mathrm{sn}^8-(1+k^2)\,\mathrm{sn}^{10}+\mathrm{sn}^{12} k^2,$$

$$\mathrm{sn}^2\,\mathrm{cn}^4\,\mathrm{dn}^4 = \mathrm{sn}^2-2\mathrm{sn}^4(k^2+1)+\mathrm{sn}^6(k^4+4k^2+1)-2\mathrm{sn}^8 k^2(1+k^2)+\mathrm{sn}^{10}k^4,$$

$$\mathrm{sn}^4\,\mathrm{cn}^4\,\mathrm{dn}^4 = \mathrm{sn}^4-2\mathrm{sn}^6(k^2+1)+\mathrm{sn}^8(k^4+4k^2+1)-2\mathrm{sn}^{10} k^2(1+k^2)+\mathrm{sn}^{12}k^4,$$

$$\mathrm{cn}^6\,\mathrm{dn}^6 = 1-3(k^2+1)\mathrm{sn}^2+\mathrm{sn}^4(9k^2+3+3k^4)-\mathrm{sn}^6[9k^2(k^2+1)+k^6+1]$$

$$+\,\mathrm{sn}^8(9k^4+3k^6+3k^2)-3\mathrm{sn}^{10}k^4(k^2+1)+\mathrm{sn}^{12}k^6,$$

$$\mathrm{sn}^2\,\mathrm{cn}^4\,\mathrm{dn}^2 = \mathrm{sn}^2-(k^2+2)\,\mathrm{sn}^4+\mathrm{sn}^6(2k^2+1)-\mathrm{sn}^8 k^2,$$

$$\mathrm{sn}^4\,\mathrm{cn}^4\,\mathrm{dn}^2 = \mathrm{sn}^4-(k^2+2)\,\mathrm{sn}^6+\mathrm{sn}^8(2k^2+1)-\mathrm{sn}^{10} k^2,$$

$$\mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{dn}^4 = \mathrm{sn}^2-(2k^2+1)\,\mathrm{sn}^4+k^2(k^2+2)\mathrm{sn}^6-k^4\,\mathrm{sn}^8,$$

$$\mathrm{sn}^4\,\mathrm{cn}^2\,\mathrm{dn}^4 = \mathrm{sn}^4-(2k^2+1)\,\mathrm{sn}^6+k^2(k^2+2)\,\mathrm{sn}^8-k^4\,\mathrm{sn}^{10}.$$

Averaging the sinus elliptic functions according to Byrd [19] one gets

$$A_2 = \int_0^{4\mathrm{K}} \mathrm{sn}^2\,\mathrm{d}\psi = \frac{4}{k^2}\,(K = E),$$

$$A_4 = \int_0^{4\mathrm{K}} \mathrm{sn}^4\,\mathrm{d}\psi = \frac{4}{3k^4}\,[(2+k^2)K-2(1+k^2)E],$$

$$A_6 = \int_0^{4\mathrm{K}} \mathrm{sn}^6\,\mathrm{d}\psi = \frac{4(1+k^2)A_4-3A_2}{5k^2}$$

$$= \frac{4}{15k^6}\,[(8+3k^2+4k^4)K-(8+7k^2+8k^2)E],$$

$$A_8 = \int_0^{4K} \mathrm{sn}^8 \, \mathrm{d}\psi = \frac{6(1+k^2)A_6 - 5A_4}{7k^2},$$

$$A_{10} = \int_0^{4K} \mathrm{sn}^{10} \, \mathrm{d}\psi = \frac{8(1+k^2)A_8 - 7A_6}{9k^2},$$

$$A_{12} = \int_0^{4K} \mathrm{sn}^{12} \, \mathrm{d}\psi = \frac{10(1+k^2)A_{10} - 9A_8}{11k^2}, \ldots,$$

$$A_{2m+2} = \int_0^{4K} \mathrm{sn}^{2m+2} \, \mathrm{d}\psi = \frac{2m(1+k^2)A_{2m} + (1-2m)A_{2m-2}}{(2m+1)k^2}.$$